

Distorted Black Holes with Charge

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Abstract

We present new solutions to the Einstein–Maxwell equations representing a class of charged distorted black holes. These solutions are static–axisymmetric and are generalizations of the distorted black hole solutions studied by Geroch and Hartle. Physically, they represent a charged black hole distorted by external matter fields. We discuss the zeroth and first law for these black holes. The first law is proved in two different forms, one motivated by the isolated horizon framework and the other using normalizations at infinity.

1 Introduction

Among the most intriguing results in general relativity are the black hole uniqueness theorems (see e.g. [1]). For Einstein–Maxwell theory in static spacetimes, the theorem ensures that the Reissner–Nordström metric is the unique black hole solution with a regular event horizon and static, asymptotically flat domain of outer communications. However, there are some obvious situations where the hypotheses of this theorem and hence its conclusions do not hold. First of all, uniqueness may fail if we consider matter other than Maxwell fields coupled to gravity. For example, it is now well known that black holes can have Yang–Mills “hair” [2, 3], i.e. there is no such uniqueness theorem for Einstein–Yang–Mills theory. Second, there are solutions in Einstein–Maxwell theory which are asymptotically flat but do not have a regular event horizon. For example, in the C-metric solutions [4, 5] the horizon contains a nodal singularity. Finally, there are static solutions which have a regular event horizon but are not asymptotically flat. They can nonetheless be physically interesting as descriptions of the near horizon geometry of isolated black holes which are distorted by the presence of far away matter [6, 7]. There are also solutions which describe a black hole immersed in a magnetic field [8]. However, in all these cases, the distorted black holes themselves do not

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carry any charge. The purpose of this paper is to present the first family of solutions to the Einstein–Maxwell equations representing distorted charged black holes. Although these charged black holes are not of direct astrophysical interest, they provide an instructive testing ground for numerous conceptual issues related to black hole mechanics and thermodynamics.

The solutions presented in this paper are a natural extension of the uncharged distorted black holes introduced in [6, 7]. In vacuum, static axi-symmetric spacetimes, Einstein’s equations reduce to Laplace’s equation on flat space. Since this equation is linear, distorted Schwarzschild black holes can be obtained by adding an appropriate distortion function to the Schwarzschild solution. This strategy cannot be extended to Einstein–Maxwell theory because we obtain a non-linear set of coupled partial differential equations. However, there exists a remarkable mapping [9, 10] which takes a static, axi-symmetric vacuum solution to a non-trivial class of static solutions in Einstein–Maxwell theory. In particular, the Schwarzschild family is mapped to the Reissner–Nordström family under this transformation. In order to obtain a class of distorted, charged black holes, we apply this transformation to the distorted Schwarzschild spacetimes of [6]. The resulting class of solutions are static and have a regular event horizon but, as in the uncharged case, they are not asymptotically flat. Therefore, fall outside the scope of the uniqueness theorems. Furthermore, we cannot introduce the notion of null infinity and consequently, the standard concept of an event horizon is not applicable. Nonetheless, these solutions can be interpreted as representing black holes via two arguments. First, they do admit locally defined isolated horizons [11, 12]. Alternatively, the solutions may be extended as asymptotically flat spacetimes by adding matter far away from the isolated horizon; this allows us to identify the isolated horizon as an event horizon. The additional matter necessarily lies outside the scope of Einstein–Maxwell theory so that, once again, the uniqueness theorems are not applicable.

The black holes presented in this paper obey the zeroth and first laws of black hole mechanics. While the zeroth law is unambiguous, we shall discuss two derivations of the first law. The first is based upon the isolated horizon framework and is intrinsically local to the horizon while the second relies on the global structure of spacetime. For static solutions in Einstein–Maxwell theory, the first laws obtained by the two methods are identical. However, even though the solutions presented in this paper are static, we obtain two quite different versions of the first law. The difference arises because the spacetime can only be extended to be asymptotically flat in the presence of matter fields.

In standard treatments of the first law, $\delta M = \frac{1}{8\pi G}\kappa\delta a + \Phi\delta Q$, one typically considers globally static (or stationary) electrovac spacetimes and small departures therefrom. It is straightforward to calculate the area a and charge Q of the horizon. However, the surface gravity κ and electric potential Φ of the horizon can only be defined unambiguously once the Killing field has been appropriately normalized at infinity. The mass of the black hole is taken to be the ADM mass of the spacetime and the variation of the mass leads to the first law. In this framework, the area and charge of the black hole are defined locally while the mass, surface gravity and electric potential refer to infinity.

The global form of the first law for distorted charged black holes is a generalization to Einstein–Maxwell theory of the first law found by Geroch and Hartle [6] in the uncharged

case. This first law contains extra terms which are interpreted as work terms due to the matter fields surrounding the black hole. However, the interpretation of the extra terms in the first law is only heuristic. Furthermore, although the black hole mass appearing in this first law satisfies a Smarr formula, it is not the ADM mass of the spacetime.

The isolated horizon framework on the other hand, only refers to quantities defined intrinsically on the horizon (see [11, 12] for detailed definitions and discussions). In particular, there could be gravitational or electromagnetic radiation in an arbitrary neighbourhood of the horizon so long as none of it crosses the horizon. This framework is applicable to the distorted, charged solutions presented here provided the additional matter fields admit a Hamiltonian description and vanish in a neighbourhood of the horizon. The first law arises as a necessary and sufficient condition for time evolution to be Hamiltonian [12]. Thus, we obtain many different first laws, one for each allowed time evolution. The isolated horizon first law takes the standard form, i.e. there are no additional work terms. Furthermore, by appealing to the Reissner–Nordström solutions we can select a preferred notion of time evolution and hence a canonical choice of black hole energy.

In section 2 we review the Weyl formalism which describes all static, axisymmetric solutions, both in vacuum and Einstein–Maxwell theory. We also describe a technique [10] which allows one to obtain a static axisymmetric solution to the Einstein–Maxwell equations from a vacuum one. In section 3 we describe the distorted black hole solutions. First we discuss the uncharged case. Then, applying the techniques introduced in section 2, we construct distorted black hole solutions with charge and discuss their properties. Finally, in section 4 we discuss the zeroth and first laws of black hole mechanics.

2 Weyl Solutions

It is well known that all static axisymmetric solutions to Einstein’s equations can be expressed in Weyl form,

$$ds^2 = -e^{2\psi} dt^2 + e^{2(\gamma-\psi)}(d\rho^2 + dz^2) + e^{-2\psi}\rho^2 d\phi^2 \quad (2.1)$$

where ψ and γ are functions of ρ and z only. Let Σ be the three dimensional Riemannian manifold orthogonal to the static Killing field. Then (ρ, z, ϕ) are coordinates on Σ . In vacuum, the field equation for ψ is:

$$\psi_{,\rho\rho} + \rho^{-1}\psi_{,\rho} + \psi_{,zz} = 0. \quad (2.2)$$

Let us introduce a fictitious flat metric, h_{ab} , on Σ given by $ds_h^2 = d\rho^2 + dz^2 + \rho^2 d\phi^2$. Then (2.2) is simply the Laplace equation for ψ in Σ with respect to the flat metric h_{ab} . Laplace’s equation is particularly simple to solve and has the added advantage of being linear — if ψ_1 and ψ_2 are solutions, then so is $\psi_1 + \psi_2$. This linearity will be used critically in obtaining distorted black hole solutions. Once a solution for ψ has been found, the second metric function γ is obtained by simple integration of the remaining field equations:

$$\gamma_{,\rho} = \rho[(\psi_{,\rho})^2 - (\psi_{,z})^2] \quad \text{and} \quad \gamma_{,z} = 2\rho\psi_{,\rho}\psi_{,z}. \quad (2.3)$$

The integrability of these equations is ensured by (2.2). We impose the boundary condition that $\gamma = 0$ on the $\rho = 0$ axis at all points where ψ is nonsingular; this ensures the circumference of a circle with radius r , centered on the z -axis away from material singularities will be $2\pi r$ for $r \rightarrow 0$. This condition also serves to fix the constant freedom in γ . In fact, if $\gamma = 0$ at any point p on the z -axis, then (2.3) ensures that it will vanish at all points of the axis which are connected to p . Finally, it is clear from the form of the metric (2.1) that a solution will be asymptotically flat (and the static Killing field $\partial/\partial t$ be unit timelike at infinity) if and only if ψ and γ tend to zero at infinity.

Let us now consider static axisymmetric solutions to the Einstein–Maxwell equations. By performing a gauge transformation, the electromagnetic potential can always be cast in the form

$$A = \Phi dt + \beta d\phi,$$

where Φ can be thought of as the electromagnetic potential and β as the magnetic potential. For the remainder of the paper, we shall be interested only in electric fields and will set $\beta = 0$. However, our results can be generalized to the case of non-vanishing magnetic field, for details see [13, 10]. Alternatively, solutions with non-vanishing magnetic fields can be obtained by performing a duality rotation $F \rightarrow *F$ (and keeping the spacetime metric unchanged) on a purely electric solution.

In the Einstein–Maxwell case, the metric takes the same form as before (2.1) but the field equations are now expressed in terms of the two metric functions ψ , γ and the electromagnetic potential Φ . The resulting equations are (setting $G = 1$):

$$\begin{aligned} \psi_{,\rho\rho} + \rho^{-1}\psi_{,\rho} + \psi_{,zz} &= e^{-2\psi}(\Phi_{,\rho}^2 + \Phi_{,z}^2) \\ (\rho\Phi_{,\rho}e^{-2\psi})_{,\rho} + (\rho\Phi_{,z}e^{-2\psi})_{,z} &= 0 \\ \gamma_{,\rho} &= \rho[(\psi_{,\rho})^2 - (\psi_{,z})^2 - e^{-2\psi}(\Phi_{,\rho}^2 - \Phi_{,z}^2)] \\ \gamma_{,z} &= 2\rho[\psi_{,\rho}\psi_{,z} - e^{-2\psi}\Phi_{,\rho}\Phi_{,z}]. \end{aligned} \tag{2.4}$$

As in the vacuum case, one can solve for ψ and Φ and then integrate the last two equations to find γ . However, the equation for ψ now has source terms and is no longer linear in ψ . Thus, even if there were a simple method of obtaining solutions for ψ and Φ , the nonlinearity of the equations would prevent us from “distorting” the known black hole solutions. There is, however, a simple method of obtaining a class of static, axisymmetric solutions to the Einstein–Maxwell equations from the corresponding *vacuum* solutions. This class contains all electrovac solutions in which there is a functional relationship between the gravitational potential ψ and the electromagnetic potential Φ , i.e $\psi = \psi(\Phi)$. Although this restricts the class of spacetimes under consideration, we shall see later that the Reissner–Nordström solutions are permitted. Therefore, this class will be of interest for obtaining distorted black holes with charge.

Let us now describe how to obtain a solution to the Einstein–Maxwell equations from a vacuum solution [10, 9]. Given the quadruple $(\bar{\psi}, \bar{\gamma}, C, v)$ where $\bar{\psi}$ and $\bar{\gamma}$ satisfy the vacuum Einstein equations (2.2), (2.3) and C and v are constants, we shall construct a solution

(ψ, γ, Φ) to the Einstein–Maxwell equations (2.4). First, the potential ψ is given in terms of $\bar{\psi}$, C and v as:

$$e^{-\psi} = \frac{e^{-v}}{2} \left[\left(1 + C(C^2 - 1)^{-1/2}\right) e^{-\bar{\psi}} + \left(1 - C(C^2 - 1)^{-1/2}\right) e^{\bar{\psi}} \right]. \quad (2.5)$$

Next, we turn our attention to the electromagnetic potential. If one assumes a functional relationship between ψ and Φ , it follows from the first two equations in (2.4) that

$$\frac{d^2(e^{2\psi})}{d\Phi^2} = 2. \quad (2.6)$$

It is simple to integrate this equation twice to obtain an expression for Φ ,

$$e^{2\psi} = e^{2v} - 2e^v C\Phi + \Phi^2. \quad (2.7)$$

The integration constants have been chosen such that, with ψ given by (2.5), the first two Einstein–Maxwell equations (2.4) hold. In an asymptotically flat spacetime, we require that $\psi \rightarrow 0$ at infinity. Therefore, in such spacetimes v must be set to zero in order that Φ vanishes at infinity.

Finally, we must specify the form of γ . Remarkably, given ψ and Φ from (2.5) and (2.7) respectively, the function $\bar{\gamma}$ satisfies the last two Einstein–Maxwell equations (2.4). Therefore, we set

$$\gamma = \bar{\gamma}. \quad (2.8)$$

It is straightforward to verify that (ψ, γ, Φ) is indeed a solution to the Einstein–Maxwell equations (2.4). Furthermore, the function $\bar{\psi}$ from which ψ and Φ are obtained satisfies the Laplace equation. Therefore, we have reduced the task of solving (2.4) to solving the vacuum Laplace equation for $\bar{\psi}$ and a first order equation for $\bar{\gamma}$; all other manipulations are purely algebraic.

3 Distorted Black Holes

In this section we shall obtain distorted charged black hole solutions. These solutions generalize the previously known vacuum distorted black hole solutions of [6, 7] to the case of non-vanishing electric fields. For completeness, we begin with a review of the uncharged solutions. We then describe how these solutions can be generalized to Einstein–Maxwell theory using the techniques outlined in section 2. Finally, we discuss some interesting properties of these solutions. In section 4, we shall focus on the thermodynamics of these black holes.

3.1 Distorted Schwarzschild Solutions

The exterior region of the Schwarzschild solution is static and has an axial Killing vector. Therefore it is a vacuum Weyl solution and corresponds to a specific choice of $\bar{\psi}$.¹ Interestingly, $\bar{\psi}$ for a Schwarzschild black hole of mass A is just the flat space Newtonian potential due to a rod of length $2A$ and mass A placed symmetrically on the $\rho = 0$ axis. The function $\bar{\gamma}$ is then determined uniquely from $\bar{\psi}$ by (2.3) whence we obtain,

$$\bar{\psi}_S := \frac{1}{2} \ln \left(\frac{L - A}{L + A} \right) \quad \text{and} \quad \bar{\gamma}_S := \frac{1}{2} \ln \left(\frac{L^2 - A^2}{L^2 - \eta^2} \right), \quad (3.1)$$

where L and η are functions of ρ and z given by

$$\begin{aligned} L &= (l_+ + l_-)/2 & \eta &= (l_+ - l_-)/2 \\ l_+ &= \sqrt{\rho^2 + (z + A)^2} & l_- &= \sqrt{\rho^2 + (z - A)^2}. \end{aligned} \quad (3.2)$$

In these coordinates, the horizon H is the line segment $|z| \leq A$ on the $\rho = 0$ axis. Both $\bar{\psi}$ and $\bar{\gamma}$ are regular everywhere except in the limit $\rho \rightarrow 0$ (for $|z| \leq A$) where they diverge logarithmically. This divergence is nothing more than the usual coordinate singularity at the event horizon. To see this explicitly, one can perform the following coordinate transformation from the Weyl coordinates (t, ρ, z, ϕ) to the standard (t, r, θ, ϕ) coordinates for Schwarzschild spacetime,

$$\begin{aligned} r &= L + A & \cos \theta &= (l_+ - l_-)/2A \\ z &= L \cos \theta & \rho^2 &= (L^2 - A^2) \sin^2 \theta. \end{aligned} \quad (3.3)$$

The metric takes the usual Schwarzschild form with the line segment H mapped to the event horizon $r = 2A$. Since the Weyl coordinates are valid only in a region where the Killing vector t^a is timelike, these coordinates cover only the exterior region of the Schwarzschild solution. However, one can use the standard methods (see, for example, chapter 6 of [14]) to extend the spacetime inside the horizon and show that there is no singularity at H .

We shall now construct the distorted black hole solutions. Recall that the field equation (2.2) is linear in the function $\bar{\psi}$. This enables us to add any harmonic function $\bar{\psi}_D$ to the Schwarzschild potential $\bar{\psi}_S$ and obtain a new solution $\bar{\psi} = \bar{\psi}_S + \bar{\psi}_D$ to (2.2); this new solution may be considered to be a distorted version of the Schwarzschild solution. Furthermore, if $\bar{\psi}_D$ is regular everywhere, the new potential $\bar{\psi}$ will also be logarithmically divergent at the line segment H and thus the location of the horizon will be unchanged. However, since $\bar{\psi}_D$ satisfies Laplace's equation and is regular at the horizon, it will not tend to zero at infinity.

Given this $\bar{\psi}$, one can easily integrate the remaining field equations (2.3) to obtain the second metric function $\bar{\gamma}$. Although the field equations for $\bar{\gamma}$ are not linear, it is still helpful to express $\bar{\gamma}$ as the Schwarzschild function plus a distortion $\bar{\gamma}_D$ so that

$$\bar{\psi} = \bar{\psi}_S + \bar{\psi}_D \quad \text{and} \quad \bar{\gamma} = \bar{\gamma}_S + \bar{\gamma}_D. \quad (3.4)$$

¹In this subsection, we shall denote the functions corresponding to the Schwarzschild and distorted Schwarzschild solutions with an overbar i.e. $\bar{\psi}$ and $\bar{\gamma}$ to distinguish them from the distorted charged black holes introduced in section 3.2.

Recall that the function $\bar{\gamma}$ must vanish on the axis due to our boundary conditions. In the present case, the axis consists of two disjoint pieces so we must ensure that it is consistent to set $\bar{\gamma}_D = 0$ on both portions of the axis. It turns out that this condition will also place a restriction on $\bar{\psi}_D$. To see this, substitute (3.4) into the field equations (2.3) to obtain equations for $\bar{\gamma}_D$. We shall only be interested in the z -derivative of $\bar{\gamma}_D$,

$$\bar{\gamma}_{D,z} = 2\rho(\bar{\psi}_{S,\rho}\bar{\psi}_{D,z} + \bar{\psi}_{D,\rho}\bar{\psi}_{S,z} + \bar{\psi}_{D,\rho}\bar{\psi}_{D,z}). \quad (3.5)$$

Integrate this equation along a line parallel to and near H . Only the first term on the right hand side will contribute and, since $\bar{\psi}_{S,\rho} = 1/\rho + \mathcal{O}(1)$, it follows that $\bar{\psi}_D$ must have the same value at the two ends of H . Therefore, in order for $\bar{\gamma}$ to vanish on both disconnected sections of the axis, we require that $\bar{\psi}_D$ takes the same value \bar{u} at both ends of the line segment H . Similarly, by integrating (3.5) from one end of H to an arbitrary point on H , it follows that²

$$\bar{\gamma}_D \hat{=} 2\bar{\psi}_D - 2\bar{u}. \quad (3.6)$$

This result allows us to show that the metric is non-singular at the horizon. We shall prove this for the more general case of distorted charged black holes in the next section.

Finally, the metric can be expressed in Schwarzschild coordinates using (3.3) and the decomposition of $\bar{\psi}$ and $\bar{\gamma}$ given in (3.4). It takes the form:

$$ds^2 = -e^{2\bar{\psi}_D} \left(1 - \frac{2A}{r}\right) dt^2 + e^{2(\bar{\gamma}_D - \bar{\psi}_D)} \left[\frac{1}{(1 - 2A/r)} dr^2 + r^2 d\theta^2 \right] + e^{-2\bar{\psi}_D} r^2 \sin^2 \theta d\phi^2. \quad (3.7)$$

These are the distorted black hole solutions obtained in [6, 7]. It has been shown in [6] that the metric can be analytically continued through the horizon.

Let us now consider the behaviour of these solutions at infinity. First of all, if we assume $\bar{\psi}_D$ is harmonic everywhere and is not a constant, then it must diverge at infinity (if $\bar{\psi}_D$ is constant, by changing coordinates we can set it to zero). Therefore these solutions are not asymptotically flat. However, it is possible to find asymptotically flat extensions if we require that $\bar{\psi}_D$ is harmonic only in a neighborhood of the horizon and extend $\bar{\psi}_D$ and $\bar{\gamma}_D$ so that they tend to zero at infinity. In the intervening region we assume that $\bar{\psi}_D$ is not harmonic i.e. the vacuum Einstein equations are not satisfied in this region. In other words, there are some matter fields present in this region. This matter can be interpreted as causing the distortion of the black hole. Moreover, if the matter satisfies the strong energy condition, it follows that $\bar{u} \leq 0$.

To demonstrate that \bar{u} is non-positive, let us consider Einstein's equations projected in the t direction, as in [6]:

$$\Delta \bar{\psi}_D \equiv \bar{\psi}_{D,\rho\rho} + \rho^{-1}\bar{\psi}_{D,\rho} + \bar{\psi}_{D,zz} = 8\pi e^{2(\bar{\psi}_D - \bar{\gamma}_D)} (T_{ab} - \frac{1}{2}Tg_{ab})t^a t^b. \quad (3.8)$$

If the matter satisfies the strong energy condition, the right hand side of equation (3.8) is necessarily non-negative. Since the Laplacian is a negative operator and we have taken

²Here, and throughout the paper, $\hat{=}$ will denote equality only at H .

$\bar{\psi}_D \rightarrow 0$ at infinity, $\bar{\psi}_D$ must be non-positive everywhere in spacetime. In particular, this implies that $\bar{u} \leq 0$.

In the undistorted case, $\bar{\psi}_D = \bar{\gamma}_D = 0$, A represents the ADM mass or equivalently the Komar mass evaluated at infinity. However, this is *not* the case with non-zero distortion. The Komar mass of the spacetime as measured at infinity is not equal to A . This can be demonstrated easily by using the expression for the Komar mass:

$$M_\infty^{Komar} - M_H^{Komar} = 2 \int_\Sigma \left(T_{ab} - \frac{1}{2} T g_{ab} \right) n^a \xi^b dV \quad (3.9)$$

where Σ is a constant time spacelike hypersurface, n is the unit timelike normal to Σ and $\xi = \partial/\partial t$ is the Killing vector which is unit timelike at infinity. It is straightforward to evaluate the Komar integral at the horizon and one obtains $M_H^{Komar} = A$. We already know from the strong energy condition that the integrand on the right hand side of (3.9) is non-negative everywhere due to the strong energy condition. We can also argue that it must be positive somewhere: If it is identically zero, then from (3.8) $\bar{\psi}_D$ satisfies the vacuum equations everywhere. As argued previously, the solution cannot be vacuum everywhere *and* be asymptotically flat at infinity. Therefore, the right hand side of (3.9) is necessarily positive. Hence, we see that $M_\infty^{Komar} > M_H^{Komar} = A$. This can be understood quite easily by considering the tt component of the distorted Schwarzschild metric (3.7): $g_{tt} = e^{2\bar{\psi}_D} \left(1 - \frac{2A}{r} \right)$. The Komar mass is equal to $1/(2r^2)$ times the derivative of g_{tt} evaluated at infinity. We obtain

$$M_\infty^{Komar} = A + \lim_{r \rightarrow \infty} (r^2 \partial \bar{\psi}_D / \partial r). \quad (3.10)$$

However, from (3.8), we see that $\bar{\psi}_D$ satisfies a Laplace equation with non-zero sources. Therefore, it must tend to zero as $\mathcal{O}(1/r)$ at infinity and the second term will contribute to the Komar mass. This is consistent with the previous conclusion that $M_\infty^{Komar} > A$.

3.2 Distorted Reissner–Nordström Solutions

The vacuum-electrovac correspondence discussed in section 2 allows us to transform any given solution of the vacuum Weyl equations to a solution of the electrovac Weyl equations. It should therefore be possible to find the electrovac solutions corresponding to the distorted Schwarzschild black holes. As we shall see, these new solutions represent distorted Reissner–Nordström solutions.

The Reissner–Nordström spacetime is a static, axisymmetric solution to the Einstein–Maxwell equations. Therefore it can be cast in Weyl form with the metric functions ψ and γ given by

$$\psi_{RN} = \frac{1}{2} \ln \left(\frac{L^2 - A^2}{(L + M)^2} \right) \quad \text{and} \quad \gamma_{RN} = \frac{1}{2} \ln \left(\frac{L^2 - A^2}{L^2 - \eta^2} \right). \quad (3.11)$$

L and η are functions of ρ and z which have the same functional form as in the Schwarzschild case (3.2) and now A is given in terms of the mass M and charge Q of the black hole as

$A := \sqrt{M^2 - Q^2}$. The electric potential is given by

$$\Phi = \frac{Q}{L + M}. \quad (3.12)$$

In the limit $Q \rightarrow 0$, the electric potential becomes zero and we recover the Schwarzschild solution of mass M in Weyl coordinates. In order to recast this metric in standard Reissner–Nordström coordinates (t, r, θ, ϕ) , perform the following coordinate transformation

$$\begin{aligned} r &= L + M & \cos \theta &= (l_+ - l_-)/2A \\ z &= L \cos \theta & \rho^2 &= (L^2 - A^2) \sin^2 \theta. \end{aligned} \quad (3.13)$$

As in the Schwarzschild solution discussed in section 3.1, the horizon H is a line segment on the z -axis with $|z| \leq A$. The Weyl coordinates cover only the region outside the event horizon at $r = R \equiv M + A$. It is clear that the spacetime can be extended through the horizon in the standard way.

The functional relation (2.7) between Φ and ψ is satisfied in Reissner–Nordström spacetime with $C = M/Q$ and $v = 0$. Therefore, it must be possible to find a vacuum Weyl solution which can be transformed into the Reissner–Nordström solution. Intuitively one might expect this vacuum solution to be the Schwarzschild solution. This is indeed the case: the Schwarzschild solution with mass $A = \sqrt{M^2 - Q^2}$ is mapped to the Reissner–Nordström solution with mass M and charge Q under the transformation with $C = M/Q$ and $v = 0$. In particular,

$$\bar{\psi}_{RN} = \frac{1}{2} \ln \left(\frac{L - A}{L + A} \right), \quad \bar{\gamma}_{RN} = \frac{1}{2} \ln \left(\frac{L^2 - A^2}{L^2 - \eta^2} \right), \quad C = \frac{M}{Q} \quad \text{and} \quad v = 0. \quad (3.14)$$

Interestingly, even if we choose the constant C to be less than unity so that $\bar{\psi}$ is imaginary, the metric functions ψ and γ are still real and the metric describes a Reissner–Nordström solution with $Q > M$. We refer the reader to [10] for more details.

Since the equation for $\bar{\psi}$ is linear, we can “distort” the Reissner–Nordström solution just as we distorted the Schwarzschild solution. In particular, we can add to $\bar{\psi}_{RN}$ any regular solution $\bar{\psi}_D$ of the Laplace equation (2.2). We can then solve for $\bar{\gamma}$ and decompose it into the Reissner–Nordström function $\bar{\gamma}_{RN}$ and a distortion $\bar{\gamma}_D$. Thus, as in the distorted Schwarzschild case, we obtain

$$\bar{\psi} = \bar{\psi}_{RN} + \bar{\psi}_D \quad \text{and} \quad \bar{\gamma} = \bar{\gamma}_{RN} + \bar{\gamma}_D. \quad (3.15)$$

We can now use the vacuum-electrovac transformation discussed in (2) to find the electrovac solution corresponding to $\bar{\psi}$ and $\bar{\gamma}$. The transformation involves the two parameters C and v and we need to specify them. If $\bar{\psi}_D$ is identically zero, then we want the transformed solution to be the Reissner–Nordström solution. This means that we must choose $C = M/Q$

as before.³ As for the parameter v , recall that we chose it to be zero in section 2 because we required asymptotic flatness. However, we can no longer impose this condition since we have required $\bar{\psi}_D$ to be harmonic and regular everywhere which means that it diverges at infinity. Thus, there is no longer any reason to choose $v = 0$ and we shall just leave it as a free parameter in the transformation. (Asymptotic flatness will again be achieved by putting additional matter fields away from the black hole.)

Using (2.5) with $C = M/Q$ and v a free parameter, the metric function ψ is given (in Reissner–Nordström coordinates) by

$$e^{2\psi} = \Delta(r)e^{2\psi_D} \quad \text{where} \quad e^{-\psi_D} := e^{-v} \left[\cosh \bar{\psi}_D - \left(1 - \frac{Q^2}{Mr}\right) \frac{M}{A} \sinh \bar{\psi}_D \right]. \quad (3.16)$$

Here, we have defined $\Delta(r)$ as

$$\Delta(r) = \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right). \quad (3.17)$$

One might wonder if there is an expression similar to (3.6) relating the values of ψ_D and γ_D at the horizon. It is not obvious that the relationship $\bar{\gamma}_D \hat{=} \bar{\psi}_D - 2\bar{u}$ generalizes to the unbarred quantities. However, we do know that $\gamma = \bar{\gamma}$ everywhere. At the horizon, where $r = M + A$, the relationship (3.16) simplifies so that $\psi_D = \bar{\psi}_D + v$. Therefore,

$$\gamma_D \hat{=} 2\psi_D - 2(\bar{u} + v) \quad (3.18)$$

at the horizon of a distorted charged black hole.

Let us now express the distorted Reissner–Nordström spacetimes in terms of the familiar (t, r, θ, ϕ) coordinates (3.13). The metric is found by substituting the forms of ψ and γ given by (3.15), (3.16) into the Weyl metric. We obtain:

$$ds^2 = -\Delta(r)e^{2\psi_D}dt^2 + \frac{e^{2(\gamma_D - \psi_D)}}{\Delta(r)}dr^2 + e^{2(\gamma_D - \psi_D)}r^2d\theta^2 + e^{-2\psi_D}r^2\sin^2\theta d\phi^2. \quad (3.19)$$

To specify the solution fully, we must also give the form of the electric potential. This is found by solving (2.7) to obtain

$$\Phi = e^v \left(\frac{M}{Q} - \sqrt{\frac{M^2}{Q^2} - 1 + \Delta(r)e^{2\psi_D - 2v}} \right). \quad (3.20)$$

In order to show that these are indeed distorted charged black holes, it is necessary to demonstrate that the metric is regular at the horizon $\Delta(r) = 0$. The easiest way to analyze

³We could also choose C based on the criteria that any solution where $\bar{\psi}_D$ is everywhere constant (but not necessarily zero) should be transformed to the Reissner–Nordström solution. This leads to $C = (M \cosh \bar{u} + A \sinh \bar{u})/Q$. However, most of the results including the first law are unchanged by this choice and for the sake of simplicity, we have chosen $C = M/Q$.

this matter is to pass to ingoing Eddington-Finkelstein coordinates. Introduce a new coordinate w satisfying $dw = dt + e^{-2(\bar{u}+v)} dr/\Delta(r)$. Thus, w is similar to the standard ingoing Eddington-Finkelstein coordinate. The metric can be expressed in (w, r, θ, ϕ) coordinates as

$$ds^2 = -\Delta(r)e^{2\psi_D} dw^2 + 2e^{2\psi_D-2(\bar{u}+v)} dw dr + e^{2\psi_D} \left(\frac{e^{2\gamma_D-4\psi_D} - e^{-4(\bar{u}+v)}}{\Delta(r)} \right) dr^2 + e^{2(\gamma_D-\psi_D)} r^2 d\theta^2 + e^{-2\psi_D} r^2 \sin^2 \theta d\phi^2. \quad (3.21)$$

At first it appears that the coefficient of the dr^2 term becomes infinite at the horizon. However, (3.18) guarantees that both the numerator and denominator vanish at the horizon. One can then expand both expressions in powers of ρ^2 to show that the coefficient remains finite (but generically non-zero) at the horizon.

Since the spacetime is not asymptotically flat, it is not possible to define the concept of an event horizon (since this requires a notion of null infinity). However, there are several reasons which suggest these solutions contain black holes. Firstly, the surface $r = R := M + A$ is a Killing horizon of the Killing vector $\xi \propto \frac{\partial}{\partial w}$. Secondly, the more general notion of an isolated horizon [11, 12] is also applicable here. Clearly, all the isolated horizon conditions are satisfied here since this is a Killing horizon. These are two local justifications for calling this a black hole solution. Finally, it is not difficult to show as in [6] that this solution can be extended to be asymptotically flat, in which case the horizon will truly be the event horizon of a black hole. To do so, assume the solution presented above is valid only in a neighbourhood of the horizon. Outside this region, ψ , γ and Φ can be extended arbitrarily so that they tend to zero at spatial infinity. In the intervening region, the electrovac equations (2.4) will not be satisfied. This indicates the presence of non-electromagnetic matter in the region. Physically, one can think of this matter as causing the distortion of the black hole. As in the uncharged case, if the matter satisfies the strong energy condition it follows that $\bar{u} + v \leq 0$.

The Komar mass at the horizon for the charged solution is A and the Komar mass at infinity is now M plus the ‘ $1/r$ part of ψ_D ’. Physically, we expect ψ_D to fall off as $1/r$ and be negative whence the Komar mass at infinity must be strictly greater than M . Intuitively, this is reasonable because, as in the Reissner–Nordström solution, the difference $(M - A)$ should be the energy in the electromagnetic field whereas the $1/r$ part of ψ_D gives the energy in the additional matter fields. However, it is not clear how to make this statement more precise because we have extended the electromagnetic potential in an essentially arbitrary fashion and the energy in the electromagnetic field may not be exactly $(M - A)$.

3.3 Properties

Let us now turn our attention to some of the properties of the metric given above. In particular, we shall focus attention on the horizon $r = R := (M + A)$. The metric on a cross section of the horizon is given by

$$ds^2 = e^{2(\gamma_D-\psi_D)} R^2 d\theta^2 + e^{-2\psi_D} R^2 \sin^2 \theta d\phi^2. \quad (3.22)$$

The area of a cross section of the horizon is easily found, making use of (3.18), to be

$$a_H \cong 4\pi R^2 e^{-2(\bar{u}+v)} \quad (3.23)$$

whence the area radius of the horizon r_H is given by $r_H := R e^{-(\bar{u}+v)}$. It is already clear from (3.22) that the cross sections of the horizon are indeed distorted. To make this more explicit, we can calculate the curvature of any 2-sphere cross section of the horizon,

$${}^2R \cong \frac{e^{-2\psi_D}}{r_H^2} \left(1 - 2(\psi_{D,\theta})^2 + 3 \cot \theta \psi_{D,\theta} + \psi_{D,\theta\theta} \right). \quad (3.24)$$

Clearly, this is not constant on the 2-sphere. However, in the limit $\psi_D, \gamma_D \rightarrow 0$, the curvature tends to $1/r_H^2$ as expected for a round 2-sphere.

The electromagnetic field strength can be calculated from the potential (2.7) to be

$$\mathbf{F} = \frac{e^{2\psi_D - v}}{(A^2/Q^2 + \Delta(r)e^{2\psi_D - 2v})^{1/2}} dt \wedge \left[\Delta(r)(\psi_{D,\theta} d\theta + \psi_{D,r} dr) + \left(\frac{M}{r^2} - \frac{Q^2}{r^3} \right) dr \right] \quad (3.25)$$

which reduces to $\mathbf{F} \cong (Q/R^2)e^{2\psi_D - v} dr \wedge dt$ at the horizon. Dualizing this equation, we see that

$$\star \mathbf{F} \cong \frac{Q e^{-v}}{r_H^2} {}^2\epsilon \quad (3.26)$$

where ${}^2\epsilon = r_H^2 \sin \theta d\theta \wedge d\phi$ is the volume form on any 2-sphere cross section of the horizon. Thus, equation (3.26) implies that the “effective charge density” of the horizon is uniform. In some sense, even though the horizon is distorted, we see that the electric field at the horizon has been distorted in exactly the same manner as the geometry. Integrating (3.26) over the horizon, we find that the electric charge of the black hole is given by

$$Q_H \cong Q e^{-v}. \quad (3.27)$$

As for any isolated horizon, the solution we have presented here is of type II at the horizon. This is because the static Killing vector ξ is a repeated principal null direction of the Weyl Tensor at the horizon (see e.g. [12]). Thus in a null tetrad adapted to the horizon,⁴ the components of the Weyl tensor Ψ_0 and Ψ_1 vanish at the horizon. This implies that at the horizon, Ψ_2 is invariant under null rotations about ℓ and is given by

$$\Psi_2 \cong \frac{Q_H^2}{2r_H^4} - \frac{e^{-2\psi_D}}{2r_H^2} \left\{ 1 - 2(\psi_{D,\theta})^2 + 3 \cot \theta \psi_{D,\theta} + \psi_{D,\theta\theta} \right\}. \quad (3.28)$$

Similarly, we can explicitly show that Φ_{00} , Φ_{10} and Φ_{20} all vanish. The only relevant component of the Ricci tensor at the horizon is

$$\Phi_{11} := \frac{1}{4} R_{ab} (\ell^a n^b + m^a \bar{m}^b) = \frac{Q_H^2}{2r_H^4}. \quad (3.29)$$

⁴ ℓ is chosen parallel to the degenerate direction of the horizon, m and \bar{m} are tangent to the horizon and transverse to ℓ and n is transverse to the horizon.

4 Black Hole Thermodynamics

In this section we will discuss the zeroth and first laws of black hole mechanics for the distorted, charged black holes. We shall see that the zeroth law holds, i.e. the surface gravity of the black hole is constant. However, the form of the first law is sensitive to the choice of normalization of the timelike Killing field. We shall discuss two such choices: a local choice arising from isolated horizon considerations and a global choice of normalization at infinity. Both choices lead to a first law, but we shall see that the two versions take very different forms.

4.1 Zeroth Law

Let us first turn our attention to the surface gravity, κ . The surface gravity of a Killing horizon is typically defined as

$$\xi^a \nabla_a \xi^b = \kappa \xi^b, \quad (4.1)$$

where ξ is the horizon generating Killing vector. However, this definition does not give the surface gravity uniquely — there is a freedom to rescale ξ by a constant which results in a constant rescaling of κ . This freedom is usually fixed by appealing to infinity. In the class of spacetimes under consideration, the extension of the solution to infinity is by no means unique and will contain matter fields which we do not model. Hence, there is no natural normalization so we will only define ξ , and hence κ , up to a constant rescaling. This freedom does not affect the zeroth law: if the surface gravity is constant for one choice of ξ it is constant for every $\tilde{\xi}$ related to the original one by constant rescaling. Thus, we need only show that κ_0 corresponding to the choice $\xi_0 = \left(\frac{\partial}{\partial w}\right)$ is constant. The surface gravity is given by

$$\kappa_0 \triangleq \frac{e^{2(\bar{u}+v)}}{2R} \left(1 - \frac{Q^2}{R^2}\right) \triangleq \frac{e^{(\bar{u}+v)}}{2r_H} \left(1 - \frac{Q_H^2 e^{-2\bar{u}}}{r_H^2}\right). \quad (4.2)$$

The surface gravity κ_0 is clearly constant over the horizon. The zeroth law of black hole mechanics is satisfied.

Let us now turn our attention the value of the electromagnetic potential at the horizon. We would also expect this to be constant on the horizon of a black hole. Substituting $r = M + A$ into (3.20), it follows that

$$\Phi_H \triangleq \frac{Qe^v}{R} \triangleq \frac{Q_H e^{v-\bar{u}}}{r_H}. \quad (4.3)$$

Thus, as expected from general considerations, the electric potential is constant on the horizon. The zeroth law of black hole mechanics and “the electromagnetic version of the zeroth law” hold at the horizon of the distorted charged black holes.

4.2 Local First Law

Let us now consider a version of the first law for these black holes arising from the isolated horizon framework. In order to make the arguments in this section, we assume that the spacetime has been extended to be asymptotically flat by the addition of matter fields which admit a Hamiltonian description. Furthermore, we assume that these matter fields do not have support in a neighbourhood of the black hole, i.e. in that neighbourhood, the Einstein–Maxwell equations hold. With these assumptions, the isolated horizon results [12] are directly applicable to distorted charged black holes.

In the isolated horizon framework one attempts to find a Hamiltonian describing evolution along a vector field t^a . We assume that the magnitude of t^a is unit at infinity and at the horizon $t^a \propto \left(\frac{\partial}{\partial w}\right)^a$; this is appropriate since the horizon is non-rotating. The proportionality factor must be a constant in a given spacetime but may vary in phase space, for example it may depend upon the area and charge of the horizon and the constants \bar{u} and v ; in other words, we allow t^a to be a live vector field. Now one can ask the following question: is there a Hamiltonian H^t describing time evolution along t^a ? The answer is in the affirmative if and only if there exists a function E_H^t only of a_H and Q_H such that the first law holds:

$$\delta E_H^t = \frac{1}{8\pi} \kappa_t \delta a_H + \Phi_t \delta Q_H. \quad (4.4)$$

It turns out that E_H^t is the horizon surface term in the Hamiltonian H^t and is therefore interpreted as the horizon energy associated with time translation along t^a . Furthermore the surface gravity κ_t and electric potential Φ_t associated with t^a are also functions only of a_H and Q_H , in particular they do not depend upon the distortion parameters \bar{u} and v . We shall call t^a an admissible vector field if (4.4) is satisfied: Every admissible t^a gives rise to a first law.

It is natural to ask whether any choice of time evolution vector field t^a is ‘canonical’ in a suitable sense. For the black holes described in this paper, there is indeed a preferred choice t_0^a which is normalized appropriately at the horizon. As mentioned above, the first law (4.4) implies that both κ_t and Φ_t can be functions only of a_H and Q_H . However, there exists a two parameter family of Reissner–Nordström solutions labelled by a_H and Q_H . These solutions are in the phase space of distorted charged black holes; they simply correspond to the absence of any distorting matter. In order that the surface gravity and electric potential of the horizon take their standard values in Reissner–Nordström spacetime, we must necessarily choose t_0^a such that

$$\kappa_{t_0} \triangleq \frac{1}{2r_H} \left(1 - \frac{Q_H^2}{r_H^2}\right) \quad \text{and} \quad \Phi_{t_0} \triangleq \frac{Q_H}{r_H}. \quad (4.5)$$

This not only guarantees that κ and Φ take their usual values on Reissner–Nordström spacetimes, but also fixes their functional form for *all* distorted charged black holes: There is a unique choice t_0^a of time evolution at the horizon for which κ_{t_0} and Φ_{t_0} are given by (4.5). The energy associated with this time evolution will be denoted by $E_H^{t_0}$. By definition, $E_H^{t_0}$ satisfies the first law with κ_{t_0} and Φ_{t_0} given in (4.5). Furthermore, if we require that the

energy tends to zero as both a_H and Q_H tend to zero, it follows that

$$E_H^{t_0} = \frac{1}{4\pi} \kappa_{t_0} a_H + \Phi_{t_0} Q_H. \quad (4.6)$$

Therefore, the isolated horizon framework provides a canonical definition of the energy of a distorted black hole.

4.3 Global First Law

There is a second form of the first law for distorted charged black holes. Here we wish to consider normalizations appropriate to an observer at infinity, much as Geroch and Hartle did for the uncharged case [6]. They obtained a first law $\delta M = \frac{1}{8\pi} \kappa_H \delta a_H + M_H^{Komar} \delta \bar{u}$, where M_H^{Komar} is the Komar mass of the horizon and κ_H is the surface gravity associated with the Killing field normalized to unity at infinity. The parameter \bar{u} was interpreted as the potential due to external matter and the extra term in the first law as work done on the black hole by this matter. The parameter M is interpreted as “the mass of the black hole alone as measured at infinity” (in this case it also happens to be equal to M_H^{Komar}). More specifically, it is the first term in the expression for the Komar mass at infinity (3.10), but is certainly not the mass of the entire spacetime — there will be a second contribution from the distorting matter.

We shall now describe how to obtain a similar version of the first law for distorted charged black holes. In order to do this, we must first give a prescription by which the spacetime is to be extended to infinity. This extension is essentially arbitrary and involves unknown matter fields. First, given the metric obtained previously (3.19),

$$ds^2 = -\Delta(r) e^{2\psi_D} dt^2 + \frac{e^{2(\gamma_D - \psi_D)}}{\Delta(r)} dr^2 + e^{2(\gamma_D - \psi_D)} r^2 d\theta^2 + e^{-2\psi_D} r^2 \sin^2 \theta d\phi^2, \quad (4.7)$$

we extend this to be asymptotically flat as before by requiring ψ_D , γ_D and $\Phi \rightarrow 0$ as $r \rightarrow \infty$. This will not be possible if the Einstein–Maxwell equations are satisfied *everywhere* in spacetime. Therefore, we assume that the spacetime satisfies the Einstein–Maxwell equations in a neighbourhood of the horizon only. Outside this region, we allow other kinds of matter which we shall not model precisely. However, this matter is parametrized by the constants \bar{u} and v which affect the values of the various parameters of the horizon:

$$\begin{aligned} a_H &= 4\pi R^2 e^{-2(\bar{u}+v)} & Q_H &= Q e^{-v} \\ \kappa_H &= \frac{e^{2(\bar{u}+v)}}{2R} \left(1 - \frac{Q^2}{R^2}\right) & \Phi_H &= \frac{Q e^v}{R}. \end{aligned} \quad (4.8)$$

Here, $R = (M + A)$ and $A^2 = M^2 - Q^2$.

In order to obtain a version of the first law, one must first decide how to define the mass of the black hole. One obvious choice is the Komar mass of the horizon, $M_H^{Komar} = A \equiv (1/4\pi) \kappa_H a_H$. However, this is not a good definition of black hole mass because it does not give the correct answer for the undistorted Reissner–Nordström solutions. The Komar mass

does not include contributions from the electromagnetic “hair”, but is just the gravitational mass of the black hole. The other natural choice is M , the parameter appearing in the metric (3.21). Why should this be interpreted as the mass of the black hole? Firstly, from the definition of A , it follows that

$$M = \sqrt{(M_H^{Komar})^2 + Q^2}$$

so it clearly contains a contribution from the electromagnetic field. Furthermore, it is straightforward to show that M satisfies a Smarr formula: $M = (1/4\pi)\kappa_H a_H + \Phi_H Q_H$. Therefore, one may interpret it as the mass of the horizon including contributions from both the gravitational and electromagnetic fields. Thus, it seems reasonable to interpret M as the total mass of the black hole while M_H^{Komar} represents only the gravitational contribution to the mass.

Now we have decided that the mass of the black hole is M , one can simply vary the above expressions for a_H and Q_H to obtain the algebraic identity:

$$\delta M = \frac{1}{8\pi}\kappa_H \delta a_H + \Phi_H \delta Q_H + \sqrt{(M_H^{Komar})^2 + Q^2} \delta v + M_H^{Komar} \delta \bar{u}. \quad (4.9)$$

This is the first law applicable for an observer at infinity. As usual, all terms on the right hand side are evaluated at the horizon (although the normalizations of t and Φ are determined at infinity). This form of the first law (4.9) is similar to that obtained in the uncharged case. In the charged case there are now two extra work terms which must be interpreted. Before doing so, let us discuss the parameters \bar{u} and v . First of all, note that the quantity \bar{u} comes directly from the distorted Schwarzschild solution and therefore it tells us how the *uncharged* part of the external matter affects the hole. Another way to see this is to notice that in (4.8), \bar{u} only affects the values of gravitational parameters at the horizon, κ_H and a_H , but not the electromagnetic ones. On the other hand, v affects both the gravitational and electromagnetic parameters. Therefore, we can think of v as describing the effective potential due to some charged matter present in the spacetime.

Let us now return to the first law. The first two terms on the right hand side of (4.9) are the usual ones describing changes in mass of the black hole due to changes in area and charge. The third term is a work term explaining how the mass of the black hole changes as we change v . We have argued above that v is a potential due to some charged matter in the spacetime. Therefore, it seems reasonable that v couples to the total mass $\sqrt{(M_H^{Komar})^2 + Q^2}$ which contains contributions from both gravitational and electromagnetic fields. In contrast, \bar{u} represents the uncharged matter in the spacetime, so it only couples to the gravitational part of the mass, M_H^{Komar} . Thus, the $M_H^{Komar} \delta \bar{u}$ term can be interpreted as the gravitational work done on the black hole by variations in the uncharged external matter while the $\sqrt{(M_H^{Komar})^2 + Q^2} \delta v$ is the work done by charged external matter.

These considerations are, of course, only heuristic. More importantly, although we have argued that M should be interpreted as the total mass of the black holes since it satisfies a Smarr formula, its physical significance is in fact not clear. It is certainly not the ADM mass

of spacetime. Due to the arguments given at the end of section 3.1, the ADM mass must also contain a contribution from ψ_D . In the distorted Schwarzschild case, M is the Komar mass at the horizon but even this is not true in the charged case. As in [6], we may perhaps think of M as “the mass of the black hole alone as measured by an observer at infinity”; i.e. it is the quantity obtained by ignoring all contributions pertaining to ψ_D in the expression for the ADM mass.

5 Conclusion

In this paper we have obtained the first family of distorted charged black hole solutions. These solutions generalize the distorted Schwarzschild solutions studied in [6, 7] to Einstein–Maxwell theory. In order to obtain black holes with non-zero charge, we have used the vacuum-electrovac correspondence found in [9, 10] to transform the distorted Schwarzschild solutions to distorted Reissner–Nordström solutions. These solutions are regular at the horizon, but are not asymptotically flat unless one includes additional matter fields in the exterior portion of spacetime. The black holes are clearly distorted since the 2-curvature of the horizon is not constant.

We have also studied the zeroth and first laws of black hole mechanics. The surface gravity and electric potential are both constant on the horizon as expected. However, when considering the first law, one must introduce a normalization of the vector field generating the horizon and also an electromagnetic gauge choice. We have presented two different choices. The first is motivated by isolated horizons and leads to the standard form of the first law. Furthermore, it is natural to normalize the vector field generating time translation such that the surface gravity and electric potential have the same functional form as in Reissner–Nordström spacetimes. This framework also leads to a canonical definition of the energy of the black hole.

The second alternative is to normalize the Killing vector and electric potential at infinity. This leads to formulae for the surface gravity and electric potential which depend upon two extra parameters, \bar{u} and v . These can be thought of as describing in some sense the amounts of uncharged and charged matter in spacetime. We have also obtained a first law tailored to this choice. As one might expect, the first law contains extra terms involving \bar{u} and v which can be thought of as work terms involving the uncharged and charged matter respectively. It is important to emphasize that the interpretations given for the global first law are all somewhat heuristic. In particular the parameter M which is to be interpreted as the mass of the black hole is not equal to the ADM mass of spacetime. Therefore, we feel that the isolated horizon framework provides a clearer interpretation of the first law for these black holes.

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